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Constructive Method for Polynomial Extensions in Two Dimensions

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Abstract

Polynomial extensions play a vital role in numerical analysis of some high-order approximation, such as the *p* version and the *h-p* version of the finite element method and spectral methods. In this paper, we construct explicitly polynomial extensions on a triangle T and a square S, which lift a polynomial defined on a side Γ or on whole boundary of T or S. These extension operators from $H_{00}^{1/2}(\Gamma)$ to $H^1(T)$ or $H^1(S)$ and from $H^{1/2}(\partial T)$ to $H^1(T)$ or from $H^{1/2}(\partial S)$ to $H^1(S)$ are continuous.

Keywords: finite element method, polynomial extension, continuous operator,convolution, compatible. Sobolev spaces

Nomenclature

p	$p \geq 0$ Integer
C	A constant independent of f and p
$P_p^1(\Omega)$	The sets of polynomials of total degree $\leq p$ on domain Ω
$P_p^2(\Omega)$	The sets of polynomials of separate degree $\leq p$ on domain Ω
$P_p(\Gamma_i)$	The polynomial space of degree $\leq p$ over Γ_i

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Γ_i	The side of the triangle $T, 1 \le i \le 3$

1. Introduction

In analysis of the high-order finite element method (FEM), such as the *p* and *h-p* versions of FEM and the spectral element method, we need to construct a globally continuous and piecewise polynomial which has the optimal estimation for its approximation error and satisfies homogeneous or non-homogeneous Dirichlet boundary conditions. The construction of such a polynomial is started with local polynomial projections on each element for the best approximation. A simple union of local polynomials projections is not globally continuous and does not satisfy the Dirichlet boundary conditions. For the continuous Galerkin method in two and three dimensions, we have to adjust these local polynomial projections by a special technique called polynomial extension or lifting. Hence, it is essential for us to build a polynomial extension compatible to FEM subspaces, by which the union of local polynomial projections can be modified to a globally continuous polynomial without degrading the best order of approximation error. The polynomial extensions together with local projections have led to the best estimation in the approximation error for the p FEM [3, 4, 5, 6,10] and *h-p* FEM and the best condition number for the preconditioning of the *p* FEM [4] and the h-p FEM [11].

Babuška and Suri [6] proposed an extension F on an equilateral triangle T = $\{(x, y) \mid -\frac{y}{\sqrt{3}} \le x \le \frac{y}{\sqrt{3}},$ $0 \le y \le \frac{\sqrt{3}}{2}\}$ with I = (0, 1) as one of its sides,

$$F^{[f]}(x,y) = \frac{\sqrt{3}}{2y} \quad = \int_{-\infty}^{+\infty} f(t) H(x-t,y)dt = (f * H(\cdot,y))(x)$$

with the characteristic function $H(x,y) = \frac{\sqrt{3}}{2y}$ for $-\frac{y}{\sqrt{3}} \le x \le \frac{y}{\sqrt{3}}$, and $H(x,y)=0$, otherwise. This extension realizes a continuous mapping $H^{1/2}(I) \to H^1(T)$ such that $F^{[f]} \in P_p^1(T)$ for $f \in P_p(I)$ and $F^{[f]}|_I$ = f. Using this extension operator of convolution type they were able to prove implicitly the existence of the continuous extension operator $R: H_{00}^{1/2}(I) \to H^1(T)$ [2, 6] such that $R^{[f]} \in P_p^1(T)$ for $f \in P_p^0(I) = \{\phi \in P_p(I) \mid \phi(0) = \phi(1) = 0\}$, and $R^{[f]}|_I = f, R^{[f]}|_{\partial T \setminus I} = 0$. Incorporating the operator R on the triangle T and a bilinear mapping of a standard square $S = (-1, 1)^2$ onto a truncated triangle $\tilde{T}$ (a trapezoid), they generalized the polynomial extension R to the square S, which realizes a continuous mapping $H_{00}^{1/2}(\Gamma) \to H^1(S)$ such that $R^{[f]} \in P_p^2(S)$ for $f \in P_p^0(I)$, and $R^{[f]}|_I = f, R^{[f]}|_{\partial S \setminus I} = 0$.

In 1991 Babuška et al.[2] proved implicitly the existence of continuous operator $E: H^{1/2}(\partial\Omega) \to H^1(\Omega), \Omega = T$ or S based on the polynomial extension F of the convolution type such that $E^{[f]} \in P_p^2(\Omega)$ for $f \in P_p(\partial\Omega)$, and $E^{[f]}|_{\partial\Omega} = f$. This polynomial extension from the whole boundary to the interior of the standard domain was utilized for preconditioning of the *p* FEM [2].

The proof of the continuity of the operator F in [2, 6] is direct and straightforward by the Fourier Transform because the operator F is defined explicitly. On the contrary the proof for the operators R and E are not straightforward and constructive since they are not explicitly constructed. Recently we have noted that the polynomial extension of convolution type has been successfully generalized to tetrahedrons [24] in three dimensions. Muňoz-Sola creatively developed the polynomial extension of convolution type on a tetrahedron by constructing explicitly the extension operator R, for which the proof for the continuity is explicit and constructive. Although the polynomial extension on a tetrahedron can not be generalized to construct extensions on prisms and hexahedrons, we found that the structure of the extension operator R introduced there can be adopted for the polynomial extension R on a triangle in two dimensions, which can make the proof constructive and straightforward. In this paper we would explicitly construct the operators R and E on a triangle T and a square S.

2. Polynomial extension on a triangle

2.1. Polynomial extension R from one side of a triangle

Let $T=\{(x,y)\,|\,0\le y\le 1-x, 0<x<1\}$ be a right triangle in $\mathbb{R}^2$, $f\in L^2_\beta(\Gamma_1)=\{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}f(x)\in L^2(\Gamma_1)\}, \Gamma_1=I=[0,1], \beta=1/2.$ We define an operator F by

$$F^{[f]}(x,y)=\frac{1}{y}\int_x^{x+y} f(\xi)d\xi \tag{1}$$

Obviously, $F^{[f]}(x,0)=f(x),$ and $F^{[f]}(x,y)\in P^1_p(T)$ if $f\in P_p(\Gamma_1).$

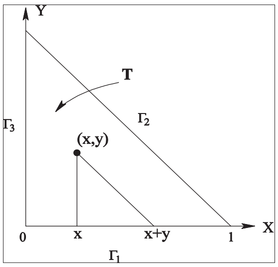

Fig.1. A standard triangular domain T

Lemma 1 Let $F^{[f]}$ is defined as in (1). Then for $f\in L^2_\beta(\Gamma_1)$ there hold

$$\| F^{[f]}(x,y)\|_{L^2(T)}\le C\| x^{\frac{1}{2}}f(x)\|_{L^2(\Gamma_1)} \tag{2}$$

and

$$\| F^{[f]}(x,y)\|_{L^2(T)}\le C\| (1-x)^{\frac{1}{2}}f(x)\|_{L^2(\Gamma_1)} \tag{3}$$

where C is a constant independent of f and p.

Theorem 1 Let F be the operator defined as in (1). Then F is a continuous mapping: $H^t(\Gamma_1)\to H^{t+\frac{1}{2}}(T), t=0,\frac{1}{2}$ such that $F^{[f]}(x,y)|_{\Gamma_1}=f(x)$ for $f(x)\in H^t(\Gamma_1),$

$$\| F^{[f]}(x,y)\|_{H^1}(T)\le C\| f(x)\|_{H^{\frac{1}{2}}(\Gamma_1)},$$

and

$$\| F^{[f]}(x,y)\|_{H^{\frac{1}{2}}(T)}\le C\| f(x)\|_{L^2(\Gamma_1)}.$$

We further introduce an operator R_T by

$$R_T^{[f]}(x,y)=x(1-x-y)F^{[\frac{f}{\xi(1-\xi)}]}(x,y)=\frac{x(1-x-y)}{y}\int_x^{x+y}\frac{f(\xi)}{\xi(1-\xi)}d\xi. \tag{4}$$

For all $f \in H_{00}^{\frac{1}{2}}$. Obviously, $R_T^{[f]}(x,y)|_{\Gamma_1} = f(x)$ and $R_T^{[f]}(x,y)|_{\Gamma_2 \cup \Gamma_3} = 0$.

Lemma 2 Let $f(x) \in H_{00}^{\frac{1}{2}}(\Gamma_1)$, and let $R_T^{[f]}(x,y)$ be given as in (4). Then

$$\| R_T^{[f]}(x,y) \|_{L^2(T)} \le C \| x^{\frac{1}{2}} f(x) \|_{L^2(\Gamma_1)} \tag{5}$$

and

$$\| R_T^{[f]}(x,y) \|_{L^2(T)} \le C \| (1-x)^{\frac{1}{2}} f(x) \|_{L^2(\Gamma_1)} \tag{6}$$

Lemma 3 Let $f(x) \in H_{00}^{\frac{1}{2}}(\Gamma_1)$, and let $R_T^{[f]}(x,y)$ be given as in (4). Then

$$\| \frac{\partial R_T^{[f]}(x,y)}{\partial x} \|_{L^2(T)} \le C \| f(x) \|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} . \tag{7}$$

Lemma 4 Let $f(x) \in P_p(\Gamma_1)$ vanishing at the endpoints of Γ_1, and let $R_T^{[f]}(x,y)$ be given by (4). Then

$$\| \frac{\partial R_T^{[f]}}{\partial y} \|_{L^2(T)} \le C \| f \|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} . \tag{8}$$

Theorem 1 Let $f(x) \in P_p^0(\Gamma_1)$, and let $R_T^{[f]}(x,y)$ be constructed as in (4). Then $R_T^{[f]}(x,y) \in P_p^{1,0}(T) = \{\Psi \in P_p^1(T) \mid \Psi|_{\Gamma_2 \cup \Gamma_3} = 0\} \subset P_p^1(T)$ and $R_T^{[f]}(x,y) \mid \Gamma_1 = f(x)$, and

$$\| R_T^{[f]}(x,y) \|_{H^1(T)} \le C \| f \|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} . \tag{9}$$

2.2. Polynomial extension E_T from whole boundary of a triangle

We shall construct an extension E_T of a polynomial defined on whole boundary ∂T of a triangle T, and the E_T is a continuous operator: $H^{\frac{1}{2}}(\partial T) \to H^1(T)$,which lifts a polynomial $f \in P_p(\partial T) = \{f \in C^0(\partial T), 1 \le i \le 3\}$ to the interior of T.

We introduce the Sobolev space $H_{00}^{\frac{1}{2}}(\Gamma_1, 0)$ by

$$H_{00}^{\frac{1}{2}}(\Gamma_1, 0) = \{v \in H^{\frac{1}{2}}(\Gamma_1) \mid x^{-\frac{1}{2}} v \in L^2(\Gamma_1)\} \tag{10}$$

with the norm

$$\| v \|^2_{H_{00}^{\frac{1}{2}}(\Gamma_1, 0))} = \| v \|^2_{H^{\frac{1}{2}}(\Gamma_1)} + \int_{\Gamma_1} \frac{|v(x)|^2}{x} dx .$$

Lemma 5 Let $f(x) \in H_{00}^{\frac{1}{2}}(\Gamma_1, 0)$, and let $R_1^f(x,y) = \frac{x}{y} \int_x^{x+y} \frac{f(\xi)}{\xi} d\xi$. Then

$$\| R_1^{[f]}(x,y) \|_{H^1(T)} \le C \| f(x) \|_{H_{00}^{\frac{1}{2}}(\Gamma_1, 0)} . \tag{11}$$

Theorem 2 There exists a linear operator $E_T : H^{\frac{1}{2}}(\partial T) \to H^1(T)$ such that $E_T^{[f]}|_{\partial T} = f$ and $E_T^{[f]} \in P_p(T)$ for $f \in P_p(\partial T)$, and

$$\| E_T^{[f]} \|_{H^1(T)} \le C \| f \|_{H^{\frac{1}{2}}(\partial T)} \tag{12}$$

3. Polynomial extension on a square

3.1. Polynomial extension R_S from one side of a square

We shall construct a polynomial extension R_S on a square $S=(-1,1)^2$, which maps a polynomial defined on one side of S to the interior of S, and this extension operator is

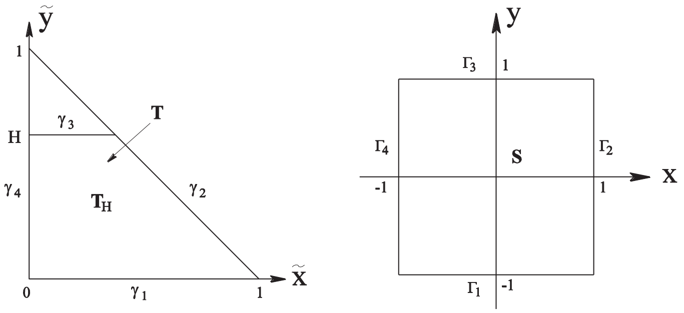

Fig.2 Mapping of square S onto

Let T_H with $0<H<1$ be a trapezoid with edge $\gamma_1,\gamma_2,\gamma_3$ and γ_4 as shown in Fig.2.where γ_2 and γ_4 are portion of the side $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ of the right triangle T, respectively. A bilinear mapping M :

maps S onto T_H and the sides Γ_i onto the sides $\gamma_i, 1\le i\le 4$.

For $f\in P_p(\Gamma_1)$, let $\tilde{f}(\tilde{x})=f(2\tilde{x}-1)=f\circ M^{-1}$, we define

$$F_S^{[f]}=F^{[\tilde{f}]}\circ M\ . \tag{13}$$

Then we have following lemma.

Lemma 6 Let $f\in H^{\frac{1}{2}}(\Gamma_1)$, and let $F_S^{[f]}$ be defined in (13). Then $F_S^{[f]}(x,y)\in P_p^2(S), F_S^{[f]}\,|\,\Gamma_1=f$ and

$$\|F_S^{[f]}\|_{H^t(S)}\le C\|f\|_{H^{t-\frac{1}{2}}(\Gamma_1)}, t=0,1 \tag{14}$$

For $f\in P_p(\Gamma_1)$ vanishing at the endpoints of Γ_1, let $\tilde{f}(\tilde{x})=f(2\tilde{x}-1)=f\circ M^{-1}, U(x,y)=R^{[\tilde{f}]}\circ M$,we introduce

$$R_S^{[f]}(x,y)=U(x,y)-U(x,1)\frac{y+1}{2} \tag{15}$$

Lemma 7 Let $\tilde{f}\in P_p^0(\gamma_1)$, and let $R^{[\tilde{f}]}(\tilde{x},\tilde{y})$ be given as in (4). Then

$$\|R^{[\tilde{f}]}\|_{H^1(\gamma_3)}\le C\|\tilde{f}\|_{H_{00}^{\frac{1}{2}}(\gamma_1)}$$

Theorem 3 Let $f\in P_p(\Gamma_1)$ vanishes at the endpoints of Γ_1 and $R_S^{[f]}$ be defined in (15). Then $R_S^{[f]}(x,y)\in P_p^2(S), R_S^{[f]}|_{\Gamma_1}=f$ and vanishes on other sides of S, and

$$\|R_S^{[f]}\|_{H^1(S)}\le C\|f\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)}$$

3.2. Polynomial extension E_S from whole boundary of a square

We shall next to discuss the existence of a continuous extension $E_S : H^{\frac{1}{2}}(\partial S) \to H^1(S)$ which lifts $f \in P_p(\partial S) = \{\varphi \in C^0(\partial S) \mid \varphi \mid \Gamma_i = f_i \in P_p(\Gamma_i), 1 \le i \le 4\}$ to interior of S . By Schwarz inequality, we have the following lemma:

Lemma 8 For $g \in L^2(I), I = (0,1)$, there holds for $0 < h < 1$

$$\int_0^{1-h} \left| \frac{1}{h} \int_x^{x+h} g(t)dt \right|^2 dx \le C \int_0^1 x \,|\, g(x) |^2 \, dx$$

and

$$\int_0^{1-h} \left| \frac{1}{h} \int_x^{x+h} g(t)dt \right|^2 dx \le C \int_0^1 (1-x) \,|\, g(x) |^2 \, dx \; .$$

Lemma 9 Let S be square with sides Γ_1 to Γ_4 and $f \in P_p(\Gamma_1)$. Then there exists $U \in P_p^2(S)$ such that $U = f$ on the opposite side Γ_3 and

$$\| U \|_{H^1(S)} \le C \| f \|_{H^{\frac{1}{2}}(\Gamma_1)}$$

Theorem 4 Let $S = [-1,1]^2$ be the square with sides denoted by $\gamma_i, 1 \le i \le 4$, and let $f \in P_p(\partial S)$. Then there exists an extension E_S such that $E_S^{[f]} \in P_p^2(S)$ and $E_S^{[f]} |_{\partial S} = f$, and

$$\| E_S^{[f]} \|_{H^1(S)} \le C \| f \|_{H^{\frac{1}{2}}(\partial S)}$$

where C is a constant independent of f and p .

4. Conclusion

In this paper, we introduced the idea of constructive method for polynomial extensions in two dimensions and reported our some results about explicit construction of polynomial extensions in two dimensions. Furthermore, these results can be applied to *p* and *h-p* versions finite element method. Because of the limitation of the paper's length, the details of proof of theorems in previous sections will be presented in forthcoming paper [9].

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